International Journal of Theoretical Physics, Vol. 34, No. 8, 1995

Quantum Construction of General Relativistic Spacetime

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Received September 8, 1994

A construction of a model of general relativistic spacetime that arises naturally from within standard quantum theory is presented. In terms of this model all the usual structures of general relativity theory can be given a quantum-theoretic interpretation, so that the usual barriers between the two theories are absent.

The most characteristic quantity distinguishing quantum theory from classical theory is, without doubt, the *probability amplitude*

$$\langle b|a\rangle$$

the squared absolute value of which gives, in the elegantly simple formulation suggested by David Finkelstein, the probability that an initial input of state $|a\rangle$ will result, on measurement, in an outcome characteristic of state $|b\rangle$. The history of quantum theory has consisted in finding ever better methods of analyzing and computing such amplitudes.

Dirac gave us the method of using expansions of the identity operator as a sum of orthogonal projections

$$I = \sum_n |n\rangle\langle n|$$

to give a first-stage analysis:

$$\langle b|a\rangle = \langle b|I|a\rangle = \sum_n \langle b|n\rangle\langle n|a\rangle$$

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0020-7748/95/0800-1559$07.50/0

in terms of intermediate amplitudes $\langle b | n \rangle$, $\langle n | a \rangle$, etc. Then Feynman gave us a richer analysis by exploiting the fact that a product of identities is still the identity:

$$I = \prod_s I_s = \prod_s \sum_{n_s} |n_s\rangle\langle n_s| = \sum_n \prod_s |n(s)\rangle\langle n(s)| \tag{1}$$

where in the last expression we end up summing over *quantum processes* represented as products of intermediate state operators

$$\ldots |n(s)\rangle\langle n(s)| \cdot |n(s')\rangle\langle n(s')| \cdot |n(s'')\rangle\langle n(s'')| \ldots$$

labeled by index-valued functions n of the ordering parameter s. This, of course, is just a quick derivation of Feynman's basic analytical insight: that the identity operator can be expanded as a sum over quantum processes, and, as we vary the potential intermediate state expansions, all possible quantum processes appear as candidates in the expansion. Thus we have a simple derivation of Feynman's analysis by quantum processes, but the processes all occur in the infinite-dimensional complex Hilbert space structure of pure quantum theory, and so far there is no hint of any 4-dimensional spacetime structure inherent in such processes.

When, however, we use the expansion (1) to analyze a particular amplitude such as $\langle b | a \rangle$ the situation becomes much more limited:

$$\langle b | a \rangle = \sum_n \langle b | \prod_s | n(s)\rangle\langle n(s) | a \rangle \tag{2}$$

We have now a Feynman sum over amplitudes for the various intermediate processes, but we note that the only intermediate states $|n(s)\rangle$ that can possibly make a nonzero contribution are those that are *not orthogonal to either* $|a\rangle$ *or* $|b\rangle$, that is,

$$|n(s)\rangle = c_a(s)|a\rangle + c_b(s)|b\rangle$$
$$+ \text{ noncontributing components,} \qquad c_a(s) \neq 0, \quad c_b(s) \neq 0$$

To see this we only need to note that, if either $\langle b | n(s)\rangle = 0$ or $\langle n(s) | a \rangle = 0$, all sum-over-process expansions of these intermediate amplitudes vanish, and so make only a zero contribution to expression (2). Thus, in the nontrivial case where $|b\rangle$ is linearly independent of $|a\rangle$ (the only case we will consider, since otherwise $|\langle b | a \rangle| = 1$ and there is nothing to analyze), we have, for the contributing part of $|n(s)\rangle$,

$$|n_{ab}(s)\rangle = \langle a | n(s)\rangle |a\rangle + \langle a_b^{\perp} | n(s)\rangle |a_b^{\perp}\rangle \tag{3}$$

where $|a_b^{\perp}\rangle$ is the unit vector orthogonal to $|a\rangle$ in the complex 2-dimensional space $\mathcal{H}_{ab}$ spanned by state vectors $|a\rangle$ and $|b\rangle$. Thus, the observable (i.e., self-adjoint) components of any state operator $|n(s)\rangle\langle n(s)|$ making a nonzero contribution to the Feynman analysis of the amplitude $\langle b|a\rangle$ can be expressed as

$$\begin{aligned}|n_{ab}(s)\rangle\langle n_{ab}(s)| = {} & |\langle a|n(s)\rangle|^2|a\rangle\langle a| + |\langle a^\perp|n(s)\rangle|^2|a_b^\perp\rangle\langle a_b^\perp| \\ & + \langle a|n(s)\rangle\langle n(s)|a_b^\perp\rangle|a\rangle\langle a_b^\perp| \\ & + \langle a_b^\perp|n(s)\rangle\langle n(s)|a\rangle|a_b^\perp\rangle\langle a|\end{aligned}$$

or, in terms of the four obvious self-adjoint operators on $\mathcal{H}_{ab}$,

$$\begin{aligned}\sigma_0 &= (1/2)(|a\rangle\langle a| + |a_b^\perp\rangle\langle a_b^\perp|) \\ \sigma_1 &= (1/2)(|a_b^\perp\rangle\langle a| + |a\rangle\langle a_b^\perp|) \\ \sigma_2 &= (i/2)(|a_b^\perp\rangle\langle a| - |a\rangle\langle a_b^\perp|) \\ \sigma_3 &= (1/2)(|a\rangle\langle a| - |a_b^\perp\rangle\langle a_b^\perp|)\end{aligned}$$

(the standard Pauli operators on $\mathcal{H}_{ab}$), we have

$$|n_{ab}(s)\rangle\langle n_{ab}(s)| = p_{ab}(s)(\sigma_0 + \boldsymbol{\sigma}(s)) \tag{4}$$

where $p_{ab}(s) = \text{trace}(\sigma_0|n(s)\rangle\langle n(s)|) \neq 0$, and

$$\begin{aligned}\boldsymbol{\sigma}(s) &= \sin\theta(s)\cos\phi(s)\,\sigma_1 \\ &\quad + \sin\theta(s)\cos\phi(s)\,\sigma_2 + \cos\theta(s)\,\sigma_3 \\ \cos\theta(s) &= \frac{|\langle a|n(s)\rangle|^2 - |\langle a_b^\perp|n(s)\rangle|^2}{p_{ab}(s)} \\ \sin\theta(s)\cos\phi(s) &= \frac{\langle a_b^\perp|n(s)\rangle\langle n(s)|a\rangle + \langle a|n(s)\rangle\langle n(s)|a_b^\perp\rangle}{p_{ab}(s)} \\ \sin\theta(s)\sin\phi(s) &= \frac{i(\langle a_b^\perp|n(s)\rangle\langle n(s)|a\rangle - \langle a|n(s)\rangle\langle n(s)|a_b^\perp\rangle)}{p_{ab}(s)}\end{aligned}$$

Miraculously, then, we find that the only quantum processes contributing to a Feynman sum-over-processes analysis of the amplitude $\langle b|a\rangle$ are exactly those that are multiples of chains of state operators of the form

$$\ldots[\sigma_0 + \boldsymbol{\sigma}(s)][\sigma_0 + \boldsymbol{\sigma}(s')][\sigma_0 + \boldsymbol{\sigma}(s'')]\ldots$$

i.e., only those processes contribute that have links or tangent vectors that are *lightlike vectors* in the 4-space $\mathcal{S}_{ab}^4$ of self-adjoint operators on the space $\mathcal{H}_{ab} \cong C^2$, when σ_0 is given its standard interpretation as the timelike component of the three spacelike Pauli operators σ_1, σ_2, σ_3. This interpretation of the operator space $\mathcal{S}^4_{ab}$ as the Minkowski space of special relativity is, of course, not new—it has been exploited in, among others Baylis *et al.* (1992) and Marlow (1986)—but this is the first instance, to my knowledge, where it has appeared as an *intrinsic implication* of standard quantum theory.

Another surprising implication of the analysis above is that, without any explicit use of the Dirac equation for relativistic particles, we have derived one of its main consequences—the only potential paths in spacetime that connect any two states $|a\rangle$ and $|b\rangle$ are *lightlike* trajectories, and this is exactly what the Dirac equation tells us is the case even for quantum fields of nonzero mass. Any apparent slower-than-light motion is the result of an averaging over the *Zitterbewegung* inherent in the solutions of the Dirac equation (Dirac, 1967, pp. 262–263), a point treated more recently by Hestenes (1990). From the point of view we are taking here, we can say that the potentiality of processes between $|a\rangle$ and $|b\rangle$ *creates* the vacuum spacetime relating the two states as the tangent space for the possible processes connecting the states.

In this way we get a quantum operator model of *special relativistic (i.e., flat) spacetime*. The interior of the lightcone is defined by mixed states, that is, averages over the pure states defining the future lightcone surface (positive operator cone in $\mathcal{S}^4$), while the past lightcone consists of operators with negative eigenvalues, and, of course, the spacelike directions are defined by operators with mixed positive and negative eigenvalues.

But where would the *curved* spacetime of general relativity come from? The natural way to construct a "curved surface" from flat pieces is to use the pieces to put together a geodesic dome in the way pioneered by Buckminster Fuller—one would get smoothness then from ever smaller and smaller pieces. To see how this comes about from pure quantum theory, we first note that so far we have been concerned with the abstract possibility of analyzing an amplitude $\langle b|a\rangle$ in terms of the Feynman sum over quantum processes connecting the states involved in the amplitude, but as long as no actual intermediate state $|c\rangle$ is physically realized, we have no basis for including in our analysis any physical properties other than those embodied in states $|a\rangle$ and $|b\rangle$, and hence we discard all properties "orthogonal" to these states. Another way of stating this is that we work with equivalence classes of intermediate states, with two states being declared equivalent if they have the same components relative to $|a\rangle$ and $|b\rangle$, and we discard all aspects about which there exists no physical information, i.e., two states are equivalent if they project onto the same state in $\mathcal{H}_{ab}$. Our previous work above can then be seen as establishing that the set of such equivalence classes has the natural structure of spacetime $\mathcal{S}^4_{ab}$.

But, as Feynman pointed out, the situation changes radically if some particular intermediate state $|c\rangle$ is actually realized to the exclusion of other possibilities—this occurs if some means of irreversibly recording distinguishing properties of $|c\rangle$ is inserted into the process; then our analysis would look like

$$\langle b|\sum \text{(processes in } \mathcal{S}^4_{cb})|c\rangle\langle c|\sum \text{(processes in } \mathcal{S}^4_{ac})|a\rangle \tag{5}$$

where we now must deal with two 4-spaces of self-adjoint operators (i.e., spacetimes) $\mathcal{S}^4_{ac}$ and $\mathcal{S}^4_{cb}$, sharing a single common lightlike ray defined by

$$|c\rangle\langle c| = \sigma_0^{ac} + \boldsymbol{\sigma}_c^{ac} = \sigma_0^{cb} + \boldsymbol{\sigma}_c^{cb} \tag{6}$$

Experimentally preparing a definite intermediate state $|c\rangle\langle c|$ forces our analysis now to consider all lightlike paths through $|a\rangle\langle a|$ in $\mathcal{S}^4_{ac}$ that passes through $|c\rangle\langle c|$ and then continue on in $\mathcal{S}^4_{cb}$ to pass through $|b\rangle\langle b|$, and experimentally fixing another intermediate state, say $|d\rangle\langle d|$, between $|c\rangle\langle c|$ and $|b\rangle\langle b|$ just serves to replace $\mathcal{S}^4_{cb}$ with two new flat spacetime "pieces," $\mathcal{S}^4_{cd}$ and $\mathcal{S}^4_{db}$, etc.,

$$\langle b|\mathcal{S}^4_{db}|d\rangle\langle d|\mathcal{S}^4_{cd}|c\rangle\langle c|\mathcal{S}^4_{ac}|a\rangle$$

We note in this construction that only actually prepared (that is, *recorded*) intermediate states introduce new spacetime 4-planes that may intersect each other at angles other than zero, or, in other words *produce "curvature."* In between such actual recorded states, there is no additional data available to justify the construction of anything other than the flat operator 4-spaces as the proper arenas for a Feynman sum-over-process analysis, and in fact, as we have seen above, any components outside of these 4-spaces simply make a null contribution to the analysis.

We emphasize this point in order to make clear this natural connection between *matter* and *geometry* in our model. Since only *recorded* states contribute to any deviations from flatness, we get a conception of matter as simply the sum total of all quantum states that, in one way or another, have become permanently realized or recorded in our quantum universe, as opposed to those that are merely potential states. We can summarize this succinctly in the convenient dictum

Matter Is Memory

and in between the actual recorded memory of states there is only the empty vacuum of 4-space potentialities. Thus, in this model, macroscopically observable bulk matter would be constituted by incredibly convoluted "microscopic" structures of 4-space planes in the infinite-dimensional self-adjoint operator space of quantum theory, representing concentrations of permanent records of states. Presumably the macroscopic Einstein tensor and its relation to overall curvature in the universe would result from a properly defined macroscopic averaging over these quantum 4-space structures.

We have so far used terms such as "geometry," "angle," "curvature," etc., without actually specifying precisely how they are to be given meaning, but we find that, once again, quantum structure comes to our rescue: the natural geometry provided by the trace metric on self-adjoint operators induces a geometry on all our 4-space substructures. Specifically, let us define, for self-adjoint operators α, β,

$$h(\alpha, \beta) \equiv l^2 \operatorname{trace}(\alpha\beta) \tag{7}$$

where l^2 is a scale factor (measuring the physically detectable effect that Hilbert space angles and lengths exert in our observable geometric universe, possibly of the order of the Planck length squared, $\cong 10^{-70}$ m^2, but, in any case, yet to be determined). This, of course, defines a Euclidean metric, which in turn defines the standard topological and differentiable structures of our quantum spacetime. The physically useful pseudo-Riemannian metric uncovered by Einstein and so successfully used in general relativity to relate gravitation to geometry would then be given at the microscopic quantum level on all our spacetime tangent 4-planes $\mathcal{S}^4$ by

$$g_q(v, w) \equiv h(v^c, w) \tag{8}$$

where

$$v = v^0\sigma_0 + \mathbf{v}, \qquad w = w^0\sigma_0 + \mathbf{w},$$
$$\mathbf{v} = v^i\sigma_i, \text{ etc.}$$

and the conjugation v^c is the natural Clifford algebra involution defined by reversal of the vectors of the algebra, i.e. $\sigma_i^c \equiv -\sigma_i$, $i = 1, 2, 3$, so that

$$v^c \equiv v^0\sigma_0 - \mathbf{v} \tag{9}$$

(Baylis *et al.*, 1992). The macroscopically smooth (differentiable) metric of Einstein's theory, and the smooth manifold structure itself, would follow from an averaging over the nonsmooth, highly "crinkly" spacetime 4-planes of our model above, that is, $g_{\text{Einstein}} = \bar{g}_q$.

One advantage of the sort of model presented here is that *all* of the standard structure of general (and special) relativity is given a precise realization within the standard structure of quantum observables as self-adjoint operators. Thus we have the best of both worlds! The smooth 4-dimensional manifold structure of Einstein's theory is replaced by a piecewise smooth (and even piecewise *flat*) operator-valued manifold with tangent spaces spanned by the observables σ_μ on complex 2-dimensional subspaces of the Hilbert space of standard quantum theory, smooth trajectories are replaced by piecewise smooth trajectories, and, with a judicious use of *almost everywhere* (a.e.) to mean "except on the boundaries," special relativity holds on each of the flat quantum spacetime pieces.

Thus we have no new interpretational problems in either quantum theory or general relativity. Especially noteworthy is the fact that in the Heisenberg picture of quantum theory, where the initial state operator ρ_0 ($= |a\rangle\langle a|$, or possibly a mixed state) merely represents initial conditions and all evolution is included in the observables, we recover both quantum-theoretic and general relativistic results from the standard expectation value formula

$$\overline{A}(s) = \mathrm{Trace}[A(s)\rho_0]$$

with uncertainty intrinsically included by

$$\Delta A(s) = [\overline{A^2}(s) - (\overline{A}(s))^2]$$

An interesting way of interpreting the quantum processes building up the general relativistic spacetime manifolds in the construction above may be found in the recent development of the consistent-histories formulation of quantum theory. A current account with full references has been given by Saunders (1993). The present author hopes in the near future to make a more detailed formulation of the work outlined here, together with its implications, but it seems worthwhile for this preliminary version to see the light of day, if for no other reason than so that others with possibly more insight can begin to make contributions.

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